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Potentials Associated with Recurrent Markov Renewal Processes*

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SUMMARY

In [1] and more recently in [2], Chapters III and VII, Spitzer constructs potentials for a particular class of recurrent Markov chains (M.C.'s), namely, the class of recurrent random walks on the n-dimensional lattice of integers ($n = 1$ or 2). In [3], [4], and [5], Kemeny and Snell construct a potential theory for arbitrary recurrent M.C.'s. Orey [6] has characterized potential kernels for recurrent M.C.'s. The potentials and kernels studied in these papers have properties which are analogous to those defined for transient M.C.'s. This paper extends the results of Kemeny and Snell to Markov renewal processes (MRP's).

Section 1 contains a brief discussion of MRP's in general, a definition of the class of MRP's to be studied in this paper, and a summary of some needed relationships proved in other papers [7-10]. Section 2 contains some preliminary lemmas. In Section 3, a potential of functions is defined for the semi-Markov process (S-MP) associated with the MRP. The concept of a left-normal MRP is introduced and its relationship with a certain potential kernel is discussed. Section 4 contains applications of these results to the important case of a continuous parameter Markov chain (c.p.M.C.). Results are also obtained by means of "duality" for the potential of σ-finite signed measures defined for a c.p.M.C. In Section 5, a potential of functions is defined for an associated three-dimensional Markov process. Section 6 contains several miscellaneous results which have applications to previous sections.

The MRP's considered are assumed to be irreducible, recurrent, non-lattice, and to satisfy hypothesis A (see below) unless otherwise specified.

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The nonlattice assumption causes no loss in generality, the lattice case (into which Kemeny and Snell's work falls) being handled by restricting t to take on only such values as are multiples of the span.

1. Introduction

A Markov Renewal process was first defined by Pyke in [7]. It may be described intuitively as follows. A stochastic process moves from one to another of a countable number of states, usually taken for convenience to be the nonnegative integers. The successive states visited form a Markov chain. The time spent in a state is a random variable whose distribution may depend on that state and the state next to be visited. Such a process is called a semi-Markov process. The corresponding Markov Renewal process is the one which records at each time t the number of visits to each possible state in the time interval $(0, t]$.

The definition given by Pyke is constructive and defines the process only up until its first "infinity" or "explosion", i.e., only as long as there has been no more than a finite number of transitions from state to state. This definition is summarized as follows. Let $I^+ = \{0, 1, 2, \cdots\}$, let $\{a_i; i \in I^+\}$ be a sequence of numbers such that $a_i \geqslant 0$ and $\sum_i a_i = 1$, and let $\{Q_{ij}; i, j \in I^+\}$ be a family of transition mass functions satisfying $Q_{ij}(x) = 0$ for $x \leqslant 0$, $H_i(0) \leqslant 1$, and $H_i(+\infty) = 1$ where $H_i = \sum_j Q_{ij}$. (All summations, unless otherwise noted, are over I^+.) Let $\{(J_n, X_n); n \geqslant 0\}$ be a two-dimensional Markov-process defined on a complete probability space $(\Omega, \mathscr{F}, P)$ by $X_0 = 0$, $P[J_0 = i] = a_i$, and

$$P[J_n = j, X_n \leqslant x \mid J_0, X_0, \ldots, J_{n-1}, X_{n-1}] = Q_{J_{n-1},j}(x) \qquad \text{(a.s.)} \qquad (1.1)$$

for all $n \geqslant 1$. Upon setting $S_n = \sum_{i=0}^{n} X_i$, $N(t) = \sup\{n \geqslant 0 : S_n \leqslant t\}$, and $N_j(t) = \operatorname{card}\{k : 0 < k \leqslant N(t), J_k = j\}$, one obtains the process $\{\mathbf{N(t)} = (N_0(t), N_1(t), \cdots); t \geqslant 0\}$ which is called an MRP determined by $\{a_i\}$ and $\{Q_{ij}\}$. The process $\{Z_t; t \geqslant 0\}$ where $Z_t = J_{N(t)}$ is called a semi-Markov process. It is clear from these definitions that $N(t)$ counts the number of transitions in $(0, t]$, that $N_j(t)$ counts the number of transitions into state j in $(0, t]$, and that Z_t records the state being visited at time t. In [9], Pyke and Schaufele extend the definition to allow for explosions. This definition is presented here for the reader's convenience.

Let $\{Y_t = (Z_t, U_t); t \geqslant 0\}$ be any separable Markov process defined on a complete probability space $(\Omega, \mathscr{F}, P)$, having state space

$$\mathfrak{X} = \{0, 1, 2, \cdots\} \times [0, +\infty],$$

having the strong Markov property for any stopping time which is almost surely finite, and having the following properties:

(1) It has a stationary transition function $P_t(\cdot, \cdot)$ such that for each $t \geqslant 0$ and $\omega \in \mathfrak{X}$, $P_t(\omega, \cdot)$ is a probability measure on $\mathscr{B}(\mathfrak{X})$, the natural Borel field for $\mathfrak{X}$, and $P_{(\cdot)}(\cdot, A)$ is jointly measurable as a function of (t, ω) for each $A \in \mathscr{B}(\mathfrak{X})$.

(2) If $P_t(i, x; j, y) \equiv P_t[(i, x), \{j\} \times [0, y]]$, then for each fixed $t > 0$, $i, j \in I^+$ and $x \geqslant 0$, $P_t(i, x; j, \cdot)$ is a nondecreasing function which satisfies

$$P_t(i, x; j, y) = \begin{cases} P_t(i, x; j, t -) & \text{if} \quad t \leqslant y < t + x \\ P_t(i, x; j, t -) + \delta_{ij} P_t[(i, x); \{i\} \times \{t + x\}] & \\ & \text{if} \quad y \geqslant t + x. \end{cases} \tag{1.2}$$

(3) The functions $N_j(t) = \operatorname{card}\{0 < u \leqslant t : Z_{u-} \neq Z_u = j\}$ are random variables (possibly infinite) for each $j \in I^+$ and $t > 0$. Set $N_j(0) = 0$.

DEFINITION 1.1. The process $\{Z_t; t \geqslant 0\}$ obtained as the first component of the above described process is called a *semi-Markov process*, and the process $\{\mathbf{N(t)} = (N_0(t), N_1(t), ...); t \geqslant 0\}$ is called a *Markov renewal process*.

It is clear that Z_t still records the state being visited at time t and $N_i(t)$ records the number of visits to state i in $(0, t]$ by the Z-process.

DEFINITION 1.2. An MRP is said to be *regular* if $P[N_i(t) < +\infty] = 1$ for all $t \geqslant 0$ and $i \in I^+$. It is said to be *strongly-regular* if $P[N(t) < +\infty] = 1$ for all $t \geqslant 0$, where $N(t) = \sum_i N_i(t)$ is the total number of transitions from state to state in $(0, t]$.

The class of processes just defined is too large for the purposes of this paper since the S-MP and the MRP do not convey the same amount of information unless the MRP is regular. A subclass of processes, each member of which is regular, and which will be the class studied in this paper is defined as follows. Let $\{Z_t; t \geqslant 0\}$ be a separable stochastic process defined on a complete probability space $(\Omega, \mathscr{F}, P)$ having state space I^+, compactified by the addition of ∞, and having the following properties:

(*i*) Almost all sample functions are right continuous, have left limits on $[0, +\infty)$ and are such that if $Z_t = i$ $(i \neq \infty)$, there exists $\delta(t) > 0$ such that $Z_s = i$ for all $s \in (t, t + \delta(t))$, while if $Z_t = \infty$, there exists no $\epsilon > 0$ such that $Z_s = \infty$ for all $s \in (t - \epsilon, t + \epsilon)$. Further, $| Z_t - Z_{t-} | < +\infty$ for all $t \geqslant 0$. For each $t \geqslant 0$, define

$$U_t = \begin{cases} t & \text{if} \quad Z_u = Z_t \quad \text{for} \quad 0 \leqslant u \leqslant t \\ t - \sup\{u < t : Z_u \neq Z_t\} & \text{otherwise} \end{cases} \tag{1.3}$$

$$V_t = \inf\{u > t : Z_u \neq Z_t\} - t \tag{1.4}$$

$$X_t = U_t + V_t \tag{1.5}$$

$$Z_{t-} = \lim_{u \uparrow t} Z_u \tag{1.6}$$

$$Z_t^+ = Z_{t+V_t} \tag{1.7}$$

$$Z_t^- = Z_{(t-U_t)-} \tag{1.8}$$

That these quantities are random variables follows from (i) and the fact that the Z-process is separable. They have the following interpretations. U_t is the length of time the Z-process has spent in the state it occupies at time t. V_t is the length of time from t until the next transition by the Z-process. X_t is the total time spent in the state occupied at time t. Z_t^+ and Z_t^- are, respectively, the next state to be occupied after time t and the previous state occupied before time t by the Z-process.

(ii) There exists a family of real-valued functions $\{Q_{ij}(\cdot, \cdot); i, j \in I^+\}$ defined on $[0, +\infty) \times [0, +\infty)$ such that $Q_{ij}(u, \cdot)$ is a mass function for each $u \in [0, +\infty)$, $Q_{ij}(u, x) = 0$ for $x < 0$, $H_i(0, 0) \leqslant 1$ and $H_i(u, +\infty) = 1$ where $H_i(\cdot, \cdot) = \sum_j Q_{ij}(\cdot, \cdot)$, and

$$P[Z_t^+ = j, V_t \leqslant x \mid Z_t = i, U_t = u, (Z_s, U_s), 0 \leqslant s < t] = Q_{ij}(u, x). \tag{1.9}$$

(iii) The process $\{Y_t = (Z_t, U_t); t \geqslant 0\}$ is a two-dimensional, separable Markov Process having state space $\mathfrak{X}$, having the strong Markov property for all stopping times which are almost surely finite, and having a stationary transition function $P_t(\cdot; \cdot)$ satisfying (1) above and such that for all $i, j \in I^+$, $x, y \geqslant 0$, and $t > 0$,

$$P_t(i, x; j, y) = \begin{cases} \sum_k Q_{ik}(x; \cdot) * P_{(\cdot)}(k, 0; j, y)(t) & \text{if} \quad x + t > y \\ \sum_k Q_{ik}(x; \cdot) * P_{(\cdot)}(k, 0; j, +\infty)(t) + \delta_{ij}[1 - H_i(x; t)] & \\ & \text{if} \quad x + t \leqslant y. \end{cases} \tag{1.10}$$

(For any two real functions K and L such that $K(x) = L(x) = 0$ for $x < 0$,

$$K * L(t) = \int_{0-}^{t} K(t-u)\, dL(u)$$

whenever the integration is defined. For such functions K, define $K^{(0)}(t) = 0$ or 1 as $t < 0$ or $t \geqslant 0$ and $K^{(n)}(t) = K * K^{(n-1)}(t)$. All functions used in this paper are assumed to have value zero: if the argument is negative.)

It is easily verified that processes satisfying (i) through (iii) determine regular MRP's. MRP's that arise in this way will be said to satisfy hypothesis A.

Let $P_{i,0}$ denote the probability measure determined by the initial condition that $Z_0 = i$ and $U_0 = 0$, i.e. $P_{i,0}$ is the induced measure on the process started at $(i, 0)$. The following functions, to be used throughout the rest of this paper, are now defined in terms of $P_{i,0}$, the functions $P_t(i, x; j, y)$, and the functions $Q_{ij}(u; x)$.

$$Q_{ij}(t) = Q_{ij}(0, t), \qquad p_{ij} = Q_{ij}(+\infty) \tag{1.11.1}$$

$$F_{ij}(t) = \begin{cases} p_{ij}^{-1} Q_{ij}(t) & \text{if} \quad p_{ij} > 0 \\ F^{(0)}(t) & \text{otherwise} \end{cases} \tag{1.11.2}$$

$$P_{ij}(t) = P_{i,0}[Z_t = j] = P_t(i, 0; j, +\infty) \tag{1.11.3}$$

$$G_{ij}(t) = P_{i,0}[N_j(t) > 0] \tag{1.11.4}$$

$$M_{ij}(t) = E_{i,0}[N_j(t)] + \delta_{ij} \tag{1.11.5}$$

$$_kP_{ij}(t) = P_{i,0}[Z_t = j, N_k(t) = 0] \tag{1.11.6}$$

$$_kG_{ij}(t) = P_{i,0}[\text{for some } u \leqslant t, N_j(u) > 0, N_k(u) = 0] \tag{1.11.7}$$

$$_kM_{ij}(t) = E_{i,0}[N_j(T_k)]\,[1 - \delta_{jk}] + \delta_{ij} \tag{1.11.8}$$

where

$$T_k = \min(t, S_k) \qquad \text{and} \qquad S_k = \inf\{t > V_0 : Z_t = k\}$$

$$P_{ij}^{+}(t) = P_{i,0}[Z_t^{+} = j] \tag{1.11.9}$$

$$R_i(j, x; t) = P_{i,0}[Z_t = j, V_t \leqslant x] \tag{1.11.10}$$

$$R_i(j, k, x; t) = P_{i,0}[Z_t = j, Z_t^{+} = k, V_t \leqslant x] \tag{1.11.11}$$

$$_jR_i(k, m, x; t) = P_{i,0}[Z_t = k, Z_t^{+} = m, V_t \leqslant x, N_j(t) = 0]. \tag{1.11.12}$$

Let η_j, b_{ij}, and μ_{ij} denote the first moments of H_j, F_{ij}, and G_{ij}, respectively (whenever these exist).

A script letter will denote a matrix-valued function whose elements consist of doubly or singly indexed functions which use the same letter. Singly indexed functions are understood to yield diagonal matrices. For example $\mathscr{G} = (G_{ij})$ and $\mathscr{H} = (\delta_{ij} H_i)$. For any matrix-valued function $\mathscr{K}$, ${}^d\mathscr{K}$ is defined to be the diagonal of $\mathscr{K}$, namely ${}^d\mathscr{K} = (\delta_{ij} K_{ij})$. For any two matrix-valued functions $\mathscr{K}$ and $\mathscr{L}$, define the matrix convolution of $\mathscr{K}$ and $\mathscr{L}$ by $\mathscr{K} * \mathscr{L} = (\sum_k K_{ik} * L_{kj})$ whenever the sums are absolutely convergent. Matrix convolution is thus defined in terms of the convolutions of the functions which are elements of the matrices.

For any function defined by a capital letter, let the function defined by the corresponding small letter be its Laplace-Stieltjes transform, e.g.,

$$k(s) = \int_{0-}^{\infty} e^{-st}\, dK(t),$$

whenever the integral is well-defined.

For any function K, $\bar{K}$ is defined by $\bar{K}(t) = \int_0^t K(s)\, ds$ and $\tilde{K}$ is defined by $\tilde{K}(s) = \overline{K(+\infty) - K(s)}$, whenever these are well-defined. For example, $\bar{P}_{ij}(t) = \int_0^t P_{ij}(s)\, ds$ is the expected amount of time spent in state j by the Z-process until time t, given $Z_0 = i$ and $U_0 = 0$, whereas

$$\tilde{Q}_{ij}(x) = p_{ij} \int_0^x [1 - F_{ij}(u)]\, du.$$

Note that

$$\overline{K*L}(t) = \bar{K}*L(t) = K*\bar{L}(t).$$

The following relationships between the various functions defined in (1.11.1) through (1.11.12) are used in the remainder of the paper. Since the proofs of these relationships are similar, only one will actually be carried out. (For a more complete treatment see [7-9].)

$$\mathscr{P}(t) = \mathscr{Q} * \mathscr{P}(t) + (I - \mathscr{H})(t) \tag{1.12.1}$$

$$\mathscr{P}(t) = \mathscr{M} * (I - \mathscr{H})(t) \tag{1.12.2}$$

$$\mathscr{M}(t) = \mathscr{Q} * \mathscr{M}(t) + I = \mathscr{M} * \mathscr{Q}(t) + I \tag{1.12.3}$$

$$\mathscr{M}(t) = \mathscr{G} * {}^{d}\mathscr{M}(t) + I \tag{1.12.4}$$

$${}_{k}\mathscr{P}(t) = {}_{k}\mathscr{M} * (I - \mathscr{H})(t) \tag{1.12.5}$$

$$P_{ij}(t) = {}_{k}P_{ij}(t) + G_{ik} * P_{kj}(t) \tag{1.12.6}$$

$$M_{ij}(t) = {}_{k}M_{ij}(t) + G_{ik} * M_{kj}(t) \tag{1.12.7}$$

$$\mathscr{P}^{+}(t) = \mathscr{M} * (P - \mathscr{Q})(t) = \mathscr{M}(t)(P - I) + I. \tag{1.12.8}$$

$$R_i(j, k, x; t) = M_{ij} * [Q_{jk}(x + \cdot) - Q_{jk}(\cdot)](t) \tag{1.12.9}$$

$${}_{m}R_i(j, k, x; t) = {}_{m}M_{ij} * [Q_{jk}(x + \cdot) - Q_{jk}(\cdot)](t) \tag{1.12.10}$$

$$\begin{aligned} {}_{m}\bar{R}_i(j, k, x; t) &= \int_0^t {}_{m}R_i(j, k, x; s)\, ds \\ &= {}_{m}M_{ij} * \overline{[Q_{jk}(x + \cdot) - Q_{jk}(\cdot)]}(t) \end{aligned} \tag{1.12.11}$$

$${}_{m}\bar{R}_i(j, k, x; +\infty) = {}_{m}M_{ij}(+\infty) \int_0^x [Q_{jk}(+\infty) - Q_{jk}(u)]\, du. \tag{1.12.12}$$

Note that ${}_m\bar{R}_i(j, k, x; t)$ is the expected amount of time the (Z, Z^+, V)-process spends in the set $\{j\} \times \{k\} \times [0, x]$ in the interval $[0, t]$ before a visit to m, given $Z_0 = i$ and $U_0 = 0$. ${}_m\bar{R}_i(j, k, x; +\infty)$ has a similar interpretation.

To prove (1.12.9), for example, let $T_j^{(n)}$ be the time of the nth transition into state j by the Z-process $(n \geqslant 1)$. Let $T_j^{(0)} = 0$. Then

$$
\begin{aligned}
R_i(j, k, x; t) &= P_{i,0}[Z_t = j, Z_t^+ = k, V_t \leqslant x] \\
&= \sum_{n=0}^{\infty} P_{i,0}[Z_t = j, Z_t^+ = k, V_t \leqslant x, N_j(t) = n] \\
&= P_{i,0}[Z_t = j, Z_t^+ = k, V_t \leqslant x, t < T_j^{(1)}] \\
&\quad + \sum_{n=1}^{\infty} P_{i,0}[Z_t = j, Z_t^+ = k, V_t \leqslant x, T_j^{(n)} \leqslant t < T_j^{(n+1)}] \\
&= \delta_{ij} P_{j,0}[Z_t = j, Z_t^+ = k, V_t \leqslant x, t < T_j^{(1)}] \\
&\quad + \sum_{n=1}^{\infty} \int_{0-}^{t} P_{i,0}[Z_t = j, Z_t^+ = k, V_t \leqslant x, T_j^{(n+1)} > t \mid T_j^{(n)} = u] \\
&\qquad\qquad\qquad \times dP_{i,0}[T_j^{(n)} \leqslant u] \\
&= \delta_{ij} P_{j,0}[Z_0^+ = k, t < V_0 \leqslant t + x] \\
&\quad + \sum_{n=1}^{\infty} \int_{0-}^{t} P_{i,0}[Z_u^+ = k, t - u < V_u \leqslant x + t - u \mid T_j^{(n)} = u] \\
&\qquad\qquad\qquad \times dG_{ij} * G_{jj}^{(n-1)}(u) \\
&= \delta_{ij}[Q_{jk}(x + t) - Q_{jk}(t)] \\
&\quad + \int_{0-}^{t} P_{j,0}[Z_0^+ = k, t - u < V_0 \leqslant x + t - u]\, dG_{ij} * M_{jj}(u) \\
&= \int_0^t [Q_{jk}(x + t - u) - Q_{jk}(t - u)]\, d[\delta_{ij} + G_{ij} * M_{jj}(u)] \\
&= [Q_{jk}(x + \cdot) - Q_{jk}(\cdot)] * M_{ij}(t).
\end{aligned}
$$

In the proof, use was made of the strong Markov property, (1.12.5), and the fact that

$$\{Z_t = j, Z_t^+ = k, V_t \leqslant x, T_j^{(n+1)} > t\} = \{Z_u^+ = k, t - u < V_u \leqslant x + t - u\}$$

on the set where $T_j^{(n)} = u \leqslant t$.

2. Preliminary Lemmas

Whenever $\eta_j < +\infty$ for all $j \in I^+$, set $q_{ij} = (p_{ij} - \delta_{ij}) \eta_i^{-1}$ and $\mathscr{Q} = (q_{ij})$. Set

$$D_j(t) = \int_t^{+\infty} [1 - H_j(u)]\, du$$

and

$$D_{jm}(y; t) = \int_t^{+\infty} [Q_{jm}(u - y) - Q_{jm}(u)]\, du$$

$$= \int_t^{t+y} [Q_{jm}(+\infty) - Q_{jm}(u)]\, du \qquad \text{for} \qquad t \geqslant 0$$

and set

$$D_j(t) = D_{jm}(y; t) = 0 \qquad \text{for} \qquad t < 0.$$

The following lemma may be proved as in [8] for the strongly-regular case.

Lemma 2.1. *If $\eta_j < +\infty$ for all $j \in I^+$, then an irreducible, nonlattice MRP satisfying hypothesis A is recurrent if and only if $\bar{P}_{jj}(+\infty) = +\infty$ for all $j \in I^+$. If the MRP is strongly-regular, this is equivalent to $\sum_n p_{jj}^n = +\infty$ where $(p_{ij}^n) = (p_{ij})^n$.*

Lemma 2.2. *If $\eta_j < +\infty$ for all $j \in I^+$, then*

$$\eta^{-1}(I - Q) * \bar{\mathscr{P}}(t) = \eta^{-1}\bar{\mathscr{H}}(t) \tag{2.1}$$

$$-\mathscr{Q}\bar{\mathscr{P}}(+\infty) = I \qquad \textit{if the MRP is transient} \tag{2.2}$$

$$-\bar{\mathscr{P}}(t)\,\mathscr{Q} \textit{ is well defined} \tag{2.3}$$

and

$$-\bar{\mathscr{P}}(t)\,\mathscr{Q} = I - \mathscr{P}^+(t) + \mathscr{M} * \mathscr{D}(t)\,\mathscr{Q}. \tag{2.4}$$

Proof. Equation (2.1) is simply a reformulation of (1.12.1). Equation (2.2) follows from (2.1) by means of Lebesgue's monotone convergence theorem.

A typical term in $\mathscr{P}(t)\,\mathscr{Q}$ is

$$\sum_k \bar{P}_{ik}(t)\, q_{kj} = \sum_k \bar{P}_{ik}(t)\, \eta_k^{-1} p_{kj} - \bar{P}_{ij}(t)\, \eta_j^{-1}.$$

The first term on the right is positive and hence the whole sum is greater than $-\infty$. Further, by (1.12.2) and (1.12.8),

$$\sum_k \bar{P}_{ik}(t)\, \eta_k^{-1} p_{kj} \leqslant \sum_k M_{ik}(t)\, p_{kj} = P_{ij}^+(t) + M_{ij}(t) - \delta_{ij} < +\infty.$$

Finally

$$\begin{aligned}-\bar{\mathscr{P}}(t)\,\mathscr{Q}+\mathscr{P}^{+}(t)&=-\mathscr{M}*\tilde{\mathscr{H}}(t)\,\eta^{-1}(P-I)+\mathscr{M}(t)\,(P-I)+I\\&=\mathscr{M}*(\eta-\tilde{\mathscr{H}})\,(t)\,\eta^{-1}(P-I)+I\\&=I+\mathscr{M}*\mathscr{D}(t)\,\mathscr{Q}.\end{aligned}$$

The proof of the following lemma is similar to one by Kemeny and Snell [3].

LEMMA 2.3. *If $\eta_j < +\infty$ for all $j \in I^+$, then for any states i, j, k, $m \in I^+$ such that $m \neq j$,*

$$_j\bar{P}_{mk}(+\infty) + {_i\bar{P}_{jk}}(+\infty)\,{_iM_{mj}}(+\infty) = {_i\bar{P}_{mk}}(+\infty) + {_j\bar{P}_{ik}}(+\infty). \tag{2.5}$$

If $m = j$, then

$$_j\bar{P}_{ik}(+\infty) + {_i\bar{P}_{jk}}(+\infty) = {_j\bar{P}_{jk}}(+\infty)\,{_iM_{jj}}(+\infty). \tag{2.6}$$

PROOF. The right side of (2.5) is the expected amount of time the Z-process spends in state k before a visit to j through i, given $Z_0 = m$ and $U_0 = 0$.

Let X_r be the amount of time spent in k between the rth and $(r+1)$th visit to j $(r \geqslant 1)$. Then $\{X_i\}$ forms a sequence of independent, identically distributed random variables with $E(X_i) = {_j\bar{P}_{jk}}(+\infty)$ for $i \geqslant 1$. Let Y count the number of times the process visits j before visiting i. Then $E_{m,0}(Y) = {_iM_{mj}}(+\infty) - \delta_{mj}$. $X_1 + X_2 + \cdots + X_Y$ counts the amount of time the process spends in k between the first visit to j and the first visit to j through i. Making use of Wald's Fundamental Identity, the expected value of this random variable, given $Z_0 = m$ and $U_0 = 0$, is

$$_jP_{jk}(+\infty)\,[{_iM_{mj}}(+\infty) - \delta_{mj}].$$

Hence, the expected amount of time the process spends in k before a visit to j through i, given $Z_0 = m$ and $U_0 = 0$, is

$$_j\bar{P}_{mk}(+\infty) + {_j\bar{P}_{jk}}(+\infty)\,[{_iM_{mj}}(+\infty) - \delta_{mj}]$$

and the proof is complete. Using the fact that ${_k\bar{P}_{ij}}(+\infty) = {_kM_{ij}}(+\infty)\,\eta_j$ and $\eta_j < +\infty$ for all $j \in I^+$, one obtains the following

COROLLARY 2.1. *For any states, i, j, k, $m \in I^+$,*

$$_jM_{mk}(+\infty) + {_jM_{jk}}(+\infty)\,{_iM_{mj}}(+\infty) = {_iM_{mk}}(+\infty) + {_jM_{ik}}(+\infty) \qquad \text{if} \quad m \neq j. \tag{2.7}$$

$$_jM_{ik}(+\infty) + {_iM_{jk}}(+\infty) = {_jM_{jk}}(+\infty)\,{_iM_{jj}}(+\infty). \tag{2.8}$$

LEMMA 2.4. *For any states $i, j, k \in I^+$,*

$$| M_{ik}(t) - M_{jk}(t) | \leqslant {}_jM_{ik}(+\infty) + {}_iM_{jk}(+\infty). \tag{2.9}$$

$$| \bar{P}_{ik}(t) - \bar{P}_{jk}(t) | \leqslant {}_j\bar{P}_{ik}(+\infty) + {}_i\bar{P}_{jk}(+\infty). \tag{2.10}$$

PROOF. 1. $M_{ik}(t) - M_{jk}(t) = {}_jM_{ik}(t) + (G_{ij} - 1) * M_{jk}(t)$ by (1.12.7). Hence,

$$M_{ik}(t) - M_{jk}(t) \leqslant {}_jM_{ik}(t)$$

since $G_{ij} - 1 \leqslant 0$. Thus,

$$M_{ik}(t) - M_{jk}(t) \leqslant {}_jM_{ik}(+\infty) + {}_iM_{jk}(+\infty).$$

Similarly,

$$M_{jk}(t) - M_{ik}(t) \leqslant {}_jM_{ik}(+\infty) + {}_iM_{jk}(+\infty)$$

and the proof of (2.9) is complete.

2. (2.10) follows by exactly the same argument as in 1.

Set $V_i(t) = \inf\{u > t + V_t;\ Z_u = i\}$. Let ${}_iA^*_{kj}(t)$ denote the expected number of visits to j by the Z-process in $(t, V_i(t)]$, given $Z_0 = k$ and $U_0 = 0$; let ${}_iA_{kj}(t)$ denote the expected amount of time the Z-process spends in j in $(t, V_i(t)]$, given $Z_0 = k$ and $U_0 = 0$; and let ${}_iB_k(j, m, x; t)$ denote the expected amount of time the (Z, Z^+, V)-process spends in $\{j\} \times \{m\} \times [0, x]$ in $(t, V_i(t)]$, given $Z_0 = k$ and $U_0 = 0$. It is assumed in the following lemmas that $\eta_j < +\infty$ whenever needed.

LEMMA 2.5. *For all $i, j, k, m \in I^+$, $x \geqslant 0$, and $t \geqslant 0$ (except that $i \neq j$ in (2.11)),*

$$\begin{aligned} {}_iA^*_{kj}(t) = {}_iM_{kj}(+\infty) - \delta_{ki}\, {}_iM_{ij}(+\infty) \\ + {}_iM_{ij}(+\infty)\, M_{ki}(t) - M_{kj}(t). \end{aligned} \tag{2.11}$$

$${}_iA_{kj}(t) = {}_i\bar{P}_{kj}(+\infty) - \delta_{ki}\, {}_i\bar{P}_{ij}(+\infty) + {}_i\bar{P}_{ij}(+\infty)\, M_{ki}(t) - \bar{P}_{kj}(t). \tag{2.12}$$

$$\begin{aligned} {}_iB_k(j, m, x; t) = {}_i\bar{R}_k(j, m, x; +\infty) - \delta_{ki}\, {}_i\bar{R}_i(j, m, x; +\infty) \\ + {}_i\bar{R}_i(j, m, x; +\infty)\, M_{ki}(t) \\ - \bar{R}_k(j, m, x; t). \end{aligned} \tag{2.13}$$

PROOF. Let the initial condition $Z_0 = k$ and $U_0 = 0$ be given. To prove (2.11), let X_r be the random variable which counts the number of visits to j by the Z-process between its rth and $(r + 1)$th visits to $i (r \geqslant 1)$. $\{X_i\}$ forms a sequence of independent, identically distributed random variables. Let $Y = \sum_{r=1}^{N_i(t)} X_r$ and let X_0 count the number of visits to j before the first visit to i. $E(X_0 + Y)$ is the expected number of visits to j in $[0, V_i(t)]$ ·

$$E(X_0) = {}_iM_{kj}(+\infty)$$

and by Wald's Fundamental Identity,

$$E(Y) = E(X_i)\, E[N_i(t)] = {}_iM_{ij}(+\infty)\, [M_{ki}(t) - \delta_{ki}].$$

$M_{kj}(t)$ is the expected number of visits to j in $[0, t]$ and (2.11) follows.

If $i \neq j$, (2.12) and (2.13) are proved similarly to (2.11) by letting X_r record the amount of time the Z-process spends in j between the rth and $(r+1)$th visits to i to prove (2.12) and by letting X_r record the amount of time the (Z, Z^+, V)-process spends in $\{j\} \times \{m\} \times [0, x]$ between the rth and $(r+1)$th visits to i to prove (2.13). If $i = j$, then the right side of (2.12) reduces to $M_{kj} * D_j(t)$ which is the expected amount of time the process spends in state j until a transition after time t if $Z_t = j$, given $Z_0 = k$ and $U_0 = 0$. But this is ${}_jA_{kj}(t)$. A similar argument proves (2.13) when $i = j$.

LEMMA 2.6. *For all $i, j, k, m \in I^+$, $x \geqslant 0$, and $t \geqslant 0$ (except that $i \neq j$ in (2.14)),*

$$ {}_iA^*_{kj}(t) = \sum_{r\neq i} P^+_{kr}(t)\, {}_iM_{rj}(+\infty). \tag{2.14}$$

$$ {}_iA_{kj}(t) = \sum_{r\neq i} P^+_{kr}(t)\, {}_i\bar{P}_{rj}(+\infty) + M_{kj} * D_j(t). \tag{2.15}$$

$$ {}_iB_k(j, m, x; t) = \sum_{r\neq i} P^+_{kr}(t)\, {}_i\bar{R}_r(j, m, x; +\infty) + M_{kj} * D_{jm}(x, \cdot)\,(t). \tag{2.16}$$

PROOF. Since the proofs of the above three statements are similar, only that for (2.15) will be given. ${}_iA_{kj}(t)$ may be calculated by probability arguments as

$$ {}_iA_{kj}(t) = \sum_m \sum_{r\neq i} \int_{0-}^{+\infty} {}_i\bar{P}_{rj}(+\infty)\, R_k(m, r, du; t) \tag{2.17}$$

+ (the expected amount of time the Z-process spends in j from t until a transition after t if $Z_t = j$, given that $Z_0 = k$ and $U_0 = 0$).

The first term on the right of (2.17) reduces to $\sum_{r\neq i} P^+_{kr}(t)\, {}_i\bar{P}_{rj}(+\infty)$ and the second is easily evaluated as $M_{kj} * D_j(t)$.

LEMMA 2.7. *For all $i, j, k, m \in I^+$, and $x \geqslant 0$,*

$$\lim_{t\to+\infty} \{[M_{kk}(t) - M_{ik}(t)]\, {}_kM_{kj}(+\infty) + [M_{ij}(t) - M_{kj}(t)]\} = {}_kM_{ij}(+\infty). \tag{2.18}$$

$$\lim_{t\to+\infty} [{}_kA^*_{kj}(t) - {}_kA^*_{ij}(t)] = 0. \tag{2.19}$$

If $\lim_{t\to+\infty} M_{jj} * D_j(t)$ *exists finitely for all* $j \in I^+$, *then*

$$\lim_{t\to+\infty} \{[M_{kk}(t) - M_{ik}(t)]\, {}_k\bar{P}_{kj}(+\infty) + [\bar{P}_{ij}(t) - \bar{P}_{kj}(t)]\} = {}_k\bar{P}_{ij}(+\infty). \tag{2.20}$$

$$\lim_{t\to+\infty} [{}_kA_{kj}(t) - {}_kA_{ij}(t)] = 0. \tag{2.21}$$

$$\lim_{t\to+\infty} (M_{kj} - M_{ij}) * D_j(t) = 0. \tag{2.22}$$

If $\lim_{t\to+\infty} M_{jj} * D_{jm}(x, \cdot)\,(t)$ *exists finitely for all* $j,\ m \in I^+$ *and all* $x \geqslant 0$, *then*

$$\lim_{t\to+\infty} \{[M_{kk}(t) - M_{ik}(t)]\, {}_k\bar{R}_k(j, m, x; +\infty) + [\bar{R}_i(j, m, x; t) - \bar{R}_k(j, m, x; t)]\} = {}_k\bar{R}_i(j, m, x; +\infty). \tag{2.23}$$

$$\lim_{t\to+\infty} [{}_kB_k(j, m, x; t) - {}_kB_i(j, m, x; t)] = 0. \tag{2.24}$$

$$\lim_{t\to+\infty} (M_{kj} - M_{ij}) * D_{jm}(x, \cdot)\,(t) = 0. \tag{2.25}$$

PROOF. Since the proofs of (2.18), (2.20), and (2.23) and those of (2.19), (2.21), and (2.24) are similar, only proofs for (2.20) and (2.21) are given. Set

$$W_{kj}(t) = M_{kk}(t)\, {}_k\bar{P}_{kj}(+\infty) - \bar{P}_{kj}(t).$$

The left side of (2.20) becomes

$$W_{kj}(t) + {}_k\bar{P}_{ij}(t) - W_{kj} * G_{ik}(t) = [W_{kj}(t)\, G_{ik}(t) - W_{kj} * G_{ik}(t)] + [1 - G_{ik}(t)]\, W_{kj}(t) + {}_k\bar{P}_{ij}(t). \tag{2.26}$$

As $t \to +\infty$, the third term on the right converges to ${}_k\bar{P}_{ij}(+\infty)$. By Lemma 2.6,

$$W_{kj}(t) = {}_kA_{kj}(t) \leqslant {}_k\bar{P}_{jj}(+\infty) + M_{kj} * L_j(t).$$

Hence, by hypothesis, the second term converges to zero. For fixed u,

$$W_{kj}(t) - W_{kj}(t-u) = {}_k\bar{P}_{kj}(+\infty)\,[M_{kk}(t) - M_{kk}(t-u)] - [\bar{P}_{kj}(t) - \bar{P}_{kj}(t-u)].$$

The first term converges to $u\mu_{kk}^{-1}\, {}_k\bar{P}_{kj}(+\infty)$ by the Key Renewal theorem and the second term converges to $u\eta_j\mu_{jj}^{-1}$ since $P_{kj}(t) \to \eta_j\mu_{jj}^{-1}$. Hence, $\lim_{t\to+\infty} [W_{kj}(t) - W_{kj}(t-u)] = 0$ since ${}_k\bar{P}_{kj}(+\infty)\, \mu_{kk}^{-1} = \eta_j\mu_{jj}^{-1}$ if $\mu_{jj} < +\infty$ and both terms converge to zero if $\mu_{jj} = +\infty$. Hence

$$\lim_{t\to+\infty} \int_{0-}^{t} [W_{kj}(t) - W_{kj}(t-u)]\, dG_{ik}(u) = 0$$

since $G_{ik} \leqslant 1$ and (2.20) is proved. By Lemma 2.5, the left side of (2.21) may be written as

$${}_k\bar{P}_{kj}(+\infty)\,[M_{kk}(t) - M_{ik}(t)] + [\bar{P}_{ij}(t) - \bar{P}_{kj}(t)] - {}_k\bar{P}_{ij}(+\infty).$$

This converges to zero as $t \to +\infty$ by (2.20).

The proofs of (2.22) and (2.25) are similar, so only that for (2.22) will be given. Equation (2.22) follows from (2.19) and Lemma 2.6 after noting that

$${}_k\bar{P}_{rj}(+\infty) = \eta_j\, {}_kM_{rj}(+\infty).$$

Lemma 2.7 shows that if the quantities defined before Lemma 2.5 converge as $t \to +\infty$, they are, in the limit, independent of the initial condition, as expected.

3. Potentials for the Z-Process

In this section, a potential is defined for functions from I^+, the state space for the Z-process, to $(-\infty, +\infty)$. This definition is a simple extension of the definition given by Kemeny and Snell (see [3]) in the case of an M.C. Since the functions to be considered in this section are defined on I^+, they will be thought of as column vectors. For example, $f = (f_0, f_1, \cdots)'$. ("$'$" stands for the transpose of a vector or matrix.) Similarly, measures defined on I^+ will be thought of as row vectors.

It is assumed throughout this section that $\eta_j < +\infty$ for all $j \in I^+$ and that $\lim_{t\to+\infty} M_{jj} * D_j(t) < +\infty$ for all $j \in I^+$. The second assumption implies that $\lim_{t\to+\infty} M_{ij} * D_j(t)$ exists for all $i \in I^+$ and is independent of i. Let $m_i = {}_0M_{0i}(+\infty)$.

Definition 3.1. A σ-finite signed measure ν is said to be *superharmonic* if $\nu\mathcal{Q} \leqslant 0$, *subharmonic* if $-\nu$ is superharmonic, and *harmonic* if it is both superharmonic and subharmonic. For the rest of this section, π will denote the measure whose ith component is given by $\pi_i = \eta_i m_i = {}_0\bar{P}_{0i}(+\infty)$.

Definition 3.2. A real-valued function, g, defined on I^+ is said to be a *potential* if there exists a real-valued function, f, defined on I^+ such that $\pi\,|f| < +\infty$ and such that

$$g = \lim_{t\to+\infty} \bar{\mathscr{P}}(t)\,f. \tag{3.1}$$

Similar definitions could be given for superharmonic functions and potentials of measures and an analogous theory could be constructed. This will be left, however, until Section 4 where, for the important class of c.p.M.C.'s the results can be given with no additional labor.

Although it is assumed that all MRP's are recurrent, Definition 3.2 does not depend on this fact. If the MRP is, in fact, a transient c.p.M.C., Definition 3.2 reduces to the usual definition of a potential (see [11]) provided only that $\pi\,|f| < +\infty$. This is the content of the following

THEOREM 3.1. *If g is a potential with respect to a trainsient, irreducible c.p.M.C. and g is defined by a function f such that $\pi\,|f| < +\infty$, then $g = \bar{\mathscr{P}}(+\infty)f$.*

The proof of this theorem follows that of Kemeny and Snell for the case of a transient M.C. (see [3]) and it is, therefore, not given here. Since the material in this section extends that of Kemeny and Snell, some of the proofs parallel theirs and reference to these will not always be given.

In [3] Kemeny and Snell define potentials of functions, f, such that $\nu\,|f| < +\infty$ where ν is any positive superharmonic measure. Although this appears to be a weaker definition that Definition 3.2, the two definitions coincide for any strongly-regular recurrent MRP as the following lemma shows.

LEMMA 3.1. *If an MRP is recurrent, strongly-regular, and irreducible, then any positive superharmonic measure is harmonic. π is the unique positive harmonic measure for such an MRP.*

PROOF. Let $\nu > 0$ be superharmonic. Then $\nu\mathscr{Q} \leqslant 0$ implies that $\sum_i \nu_i \eta_i^{-1} p_{ij} \leqslant \nu_j \eta_j^{-1}$. Set $\mu_i = \nu_i \eta_i^{-1}$ so that $\mu P \leqslant \mu$. Set $\rho = \mu(I - P) \geqslant 0$. If $\rho > 0$, then $\mu = \mu P + \rho = \mu P^n + \rho(P + P^2 + \cdots + P^n)$ for all $n \geqslant 1$ and the second term converges to $+\infty$ as $n \to +\infty$ by Lemma 2.1 since the MRP is recurrent. Hence $\rho \equiv 0$ and $\nu\mathscr{Q} = 0$. The second result follows from Theorem 4.2 in [10].

The next lemma shows that the class of functions for which a potential may exist can be considerably reduced.

LEMMA 3.2. *If g is a potential defined by f, then $\pi f = 0$.*

PROOF. By hypothesis,

$$\lim_{t\to+\infty} \sum_j \bar{P}_{ij}(t) f_j < +\infty \qquad \text{and} \qquad \lim_{t\to+\infty} M_{ii}(t) = +\infty.$$

Hence,

$$\lim_{t\to+\infty} \sum_j \frac{f_j \bar{P}_{ij}(t)}{M_{ii}(t)} = 0.$$

Since

$$\lim_{t\to+\infty} \frac{\bar{P}_{ij}(t)}{M_{ii}(t)} = {}_i\bar{P}_{ij}(+\infty)$$

([9], Lemma 3.3) and $0 \leqslant \bar{P}_{ij}(t)/M_{ii}(t) \leqslant {}_i\bar{P}_{ij}(+\infty)$ by Lemma 2.5, it follows that the limit may be passed over the summation sign and the proof is completed by noting that $\sum_j {}_i\bar{P}_{ij}(+\infty) f_j = c \sum_j \pi_j f_j$ by Corollary 3.4 in [10] where $c > 0$ is a constant depending only on 0 and i.

Henceforth, in this section, only functions f such that $\pi \mid f \mid < +\infty$ and $\pi f = 0$ will be considered. Also, for the rest of this section, fix some state 0 which will be a reference state.

LEMMA 3.3. *Under the above assumption,*

$$_0\bar{\mathscr{P}}(+\infty) \mid f \mid < +\infty. \tag{3.2}$$

$$f = -\mathscr{Q}_0\bar{\mathscr{P}}(+\infty) f. \tag{3.3}$$

PROOF. 1.

$$\begin{aligned}\sum_j {}_0\bar{P}_{ij}(+\infty) \mid f_j \mid &\leqslant \sum_j ({}_0\bar{P}_{ij}(+\infty) + {}_i\bar{P}_{0j}(+\infty)) \mid f_j \mid \\ &= {}_0M_{ii}(+\infty) \sum_j {}_i\bar{P}_{ij}(+\infty) \mid f_j \mid < +\infty\end{aligned}$$

by (2.6) and the hypothesis that $\pi \mid f \mid < +\infty$.

2. By conditioning on the first state to be visited one may obtain

$$(I - Q) * {}_0\bar{\mathscr{P}}(t) = \tilde{\mathscr{H}}(t) - (Q_{i0} * {}_0\bar{P}_{0j}(t))$$

and hence

$$-\mathscr{Q}_0\bar{\mathscr{P}}(+\infty) = I - \eta^{-1}(p_{i0}\, {}_0\bar{P}_{0j}(+\infty)),$$

the limit following by Lebesgue's monotone convergence theorem.

Hence,

$$[-\mathscr{Q}_0\bar{\mathscr{P}}(+\infty)f]_i = f_i - p_{i0}\eta_i^{-1}\sum_j {}_0\bar{P}_{0j}(+\infty)f_j = f_i$$

since $\pi f = 0$.

COROLLARY 3.1. *If g is a potential defined by f, then*

$$g = \lim_{t\to+\infty} \bar{\mathscr{P}}(t)\,[-\mathscr{Q}_0\bar{\mathscr{P}}(+\infty)f].$$

The following theorem gives some of the properties of g whenever it exists.

THEOREM 3.2. *If g is a potential defined by f, then*

$$g = {}_0\bar{\mathscr{P}}(+\infty)f - \lim_{t\to+\infty} {}_0\mathscr{A}(t)f. \tag{3.4}$$

$$g = {}_0\bar{\mathscr{P}}(+\infty)f + g_0\mathbf{1}. \tag{3.5}$$

$$-\mathscr{Q}g = f. \tag{3.6}$$

PROOF. 1. By Lemma 2.2,

$$\bar{\mathscr{P}}(t)\,[-\mathscr{Q}] = I - \mathscr{P}^{+}(t) - \mathscr{M} * \mathscr{D}(t)\,[-\mathscr{Q}],$$

and by Corollary 3.1,

$$\begin{aligned} g &= \lim_{t\to+\infty} \bar{\mathscr{P}}(t)\,[-\mathscr{Q}_0\bar{\mathscr{P}}(+\infty)f] \\ &= \lim_{t\to+\infty} [I - \mathscr{P}^{+}(t) - \mathscr{M} * \mathscr{D}(t)\,(-\mathscr{Q})]_0\,\bar{\mathscr{P}}(+\infty)f \\ &= {}_0\bar{\mathscr{P}}(+\infty)f - \lim_{t\to+\infty} [\mathscr{P}^{+}(t)\,{}_0\bar{\mathscr{P}}(+\infty) + \mathscr{M} * \mathscr{D}(t)]\,f. \end{aligned}$$

The second term on the right of the above equation evaluated at i is equal to

$$\begin{aligned} &\lim_{t\to+\infty} \sum_j \Big[\sum_{r\neq 0} P^{+}_{ir}(t)\,{}_0\bar{P}_{rj}(+\infty) + M_{ij} * D_j(t) + P^{+}_{i0}(t)\,{}_0\bar{P}_{0j}(+\infty)\Big] f_j \\ &\qquad = \lim_{t\to+\infty} \sum_j {}_0A_{ij}(t)\,f_j \end{aligned}$$

by (2.15) and the fact that $\pi f = 0$ and (3.4) is proved.

2. $$g_i - g_0 = \sum_j {}_0\bar{P}_{ij}(+\infty) f_j - \lim_{t\to+\infty} \sum_j [{}_0A_{ij}(t) - {}_0A_{0j}(t)] f_j .$$

$$\lim_{t\to+\infty} [{}_0A_{ij}(t) - {}_0A_{0j}(t)] = 0$$

by Lemma 2.7 and

$$| {}_0A_{ij}(t) - {}_0A_{0j}(t) | \leqslant {}_0\bar{P}_{ij}(+\infty) + 2{}_0\bar{P}_{0j}(+\infty)\, {}_iM_{00}(+\infty)$$

by Lemmas 2.4 and 2.3. Hence the limit may be passed over the summation sign and (3.5) is proved.

3. $-\mathscr{Q}_0\bar{\mathscr{P}}(+\infty) f = f$ by Lemma 3.3 and $-\mathscr{Q}1 = 0$. Hence $-\mathscr{Q}g = f$ by 2.

Equation (3.6) means that $-\mathscr{Q}$ is a left inverse for the operation of forming a potential.

A potential kernel is now defined and its relationship to the potential just defined is investigated. Set

$$A_{ij} = \lim_{t\to+\infty} [M_{ii}(t)\, {}_i\bar{P}_{ij}(+\infty) - \bar{P}_{ij}(t)]$$

whenever this limit exists.

DEFINITION 3.3. Whenever it exists, $\mathscr{A} = (A_{ij})$ will be called a *potential kernel* with respect to the Z-process.

DEFINITION 3.4. An MRP is said to be *left-normal* if ${}_0\nu_j = \lim_{t\to+\infty} {}_0A_{ij}(t)$ exists for all $j \in I^+$ and some state, say 0.

The fact that ${}_0\nu_j$ (whenever it exists) is independent of the choice of i is shown in Lemma 2.7. The following theorem shows that the left-normality of an MRP and the existence of the potential kernel defined in Definition 3.3 are equivalent.

THEOREM 3.3. *${}_0\nu_j$ exists for all $j \in I^+$ if and only if A_{0j} exists for all $j \in I^+$ and ${}_0\nu_j = A_{0j}$. In this case ${}_i\nu_j$ exists for all $i, j \in I^+$ and ${}_i\nu_j = A_{ij}$.*

PROOF. The first statement is immediate in view of Lemmas 2.5 and 2.6. To prove the second statement, the reader may straightforwardly verify that

$$[M_{rr}(t)\, {}_r\bar{P}_{rj}(+\infty) - \bar{P}_{rj}(t)] - [M_{rr}(t)\, {}_r\bar{P}_{ri}(+\infty) - \bar{P}_{ri}(t)]\, {}_i\bar{P}_{ij}(+\infty)\, \eta_i^{-1}$$
$$= \bar{P}_{ri}(t)\, {}_i\bar{P}_{ij}(+\infty)\, \eta_i^{-1} - \bar{P}_{rj}(t)$$
$$= {}_0\bar{P}_{ri}(+\infty)\, {}_i\bar{P}_{ij}(+\infty)\, \eta_i^{-1} - {}_0\bar{P}_{rj}(+\infty) - {}_0A_{ri}(t)\, {}_i\bar{P}_{ij}(+\infty)\, \eta_i^{-1} + {}_0A_{rj}(t), \tag{3.7}$$

using the fact that ${}_0\bar{P}_{0j}(+\infty)\,{}_0\bar{P}_{0i}(+\infty)^{-1} = {}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1}$. Since by hypothesis, the limit on the right side of (3.7) exists, it follows that the limit on the left side exists. Setting $r = i$, the second term on the left becomes $M_{ii} * D_i(t)\,{}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1}$. This has a limit by assumption and the limit is $A_{ii}\,{}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1}$ by definition. Thus, the first term on the left must have a limit and this, by definition, is A_{ij}. That $A_{ij} = {}_i\nu_j$ again follows from Lemmas 2.5 and 2.6.

COROLLARY 3.2. *Under the hypotheses of the above theorem, for all $i, j \in I^+$,*

$$A_{ij} - {}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1}A_{ii} = {}_0\bar{P}_{ii}(+\infty)\,{}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1} - {}_0\bar{P}_{ij}(+\infty)$$
$$- {}_0\nu_i\,{}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1} + {}_0\nu_j\,. \qquad (3.8)$$

A sufficient condition that a potential exist and be expressible in terms of the potential kernel is given in the following

THEOREM 3.4. *If the MRP is left-normal and there exists a sequence $\{c_i\}$ of positive real numbers such that ${}_0A_{ij}(t) \leqslant c_j$ for all $t \geqslant 0$ and such that $\sum_i c_i\,|f_i| < +\infty$, then g exists and $g = {}_0\mathscr{P}(+\infty)f - \mathbf{1}_0\nu f$.*

PROOF. Since $\lim_{t\to+\infty} {}_0A_{ij}(t) = {}_0\nu_j$ by hypothesis and ${}_0A_{ij}(t) \leqslant c_j$, it follows that

$$\lim_{t\to+\infty} \sum_j {}_0A_{ij}(t)\,f_j = \sum_j {}_0\nu_j\,f_j < +\infty$$

and the proof is complete.

COROLLARY 3.3. *Under the assumptions of the above theorem, $g = -Af$.*

PROOF.

$$g_i = \sum_j [{}_0\bar{P}_{ij}(+\infty) - {}_0\nu_j]\,f_j$$
$$= \sum_j [-A_{ij} + {}_i\bar{P}_{ij}(+\infty)\,\eta_i^{-1}(A_{ii} + {}_0\bar{P}_{ii}(+\infty) - {}_0\nu_i)]\,f_j$$
$$= -\sum_j A_{ij}\,f_j\,,$$

making use of (3.8) and the fact that $\pi f = 0$.

Turning to the positive recurrent case, it is now shown that every such MRP is left-normal by means of the following

LEMMA 3.4. *If the MRP is positive recurrent, then* $\lim_{t\to+\infty} P_{ij}^{+}(t)$ *exists for all* $i, j \in I^+$ *and is given by* $p_j^+ = \sum_k p_{kj} b_{kj} \mu_{kk}^{-1}$.

PROOF. Let F_t and F be distribution functions defined on

$$I^+ \times I^+ \times [0, +\infty)$$

by

$$F_t(k, j, x) = \sum_{\substack{r \leqslant k \\ m \leqslant j}} R_i(r, m, x; t)$$

and

$$F(k, j, x) = \sum_{\substack{r \leqslant k \\ m \leqslant j}} p_{rm} \mu_{rr}^{-1} \int_{0-}^{x} [1 - F_{rm}(u)]\, du.$$

The summands in F_t converge to the corresponding summands in F by the Key Renewal theorem. The following argument allows the limit to be passed over the summation sign. If μ_t and μ represent the corresponding measures, then

$$\mu_t[I^+ \times I^+ \times [0, \infty)] = 1 = \mu[I^+ \times I^+ \times [0, +\infty)].$$

Fix $j \in I^+$ and let $f(k, r, x) = \delta_{jr}$. One may show, using an argument similar to that used by Kemeny and Snell (condition 2. p. 207 [3]), that

$$\lim_{t\to+\infty} \int_{\mathfrak{X}} f(k, r, x)\, d\mu_t(k, r, x) = \int_{\mathfrak{X}} f(k, r, x)\, d\mu(k, r, x)$$

where $\mathfrak{X} = I^+ \times I^+ \times [0, +\infty)$. That is,

$$\lim_{t\to+\infty} \mu_t[I^+ \times \{j\} \times [0, +\infty)] = \mu[I^+ \times \{j\} \times [0, +\infty)].$$

The left side of this equation is $\lim_{t\to+\infty} P_{ij}^{+}(t)$ and the right side is p_j^+.

THEOREM 3.5. *A positive recurrent MRP is left-normal and* $\sigma_j^2 < +\infty$ *for all* $j \in I^+$ *where*

$$\sigma_j^2 = \int_0^\infty (u - \eta_j)^2\, dH_j(u).$$

PROOF. By Lemma 3.4, $P_{ir}^{+}(t) \to p_r^+$. Also, $\sum_r P_{ir}^{+}(t) = 1 = \sum_r p_r^+$ and ${}_0\bar{P}_{rj}(+\infty) \leqslant {}_0\bar{P}_{jj}(+\infty)$. Hence, by a similar argument to that used in Lemma 3.4,

$$\lim_{t\to+\infty} \sum_{r\neq 0} P_{ir}^{+}(t)\, {}_0\bar{P}_{rj}(+\infty) = \sum_{r\neq 0} p_r^+\, {}_0\bar{P}_{rj}(+\infty) < +\infty.$$

Since $\lim_{t\to+\infty} M_{ij} * D_j(t) < +\infty$, it follows that ${}_0\nu_j$ exists. Since $\mu_{jj} < +\infty$, $\lim_{t\to+\infty} M_{ij} * D_j(t) < +\infty$, if and only if $\int_{-0}^{+\infty} D_j(t)\,dt < +\infty$ by the Key Renewal theorem and this is equivalent to $\sigma_j^2 < +\infty$.

Turning to the null-recurrent case, it is clear that an MRP is left-normal if and only if $\sum_{r\neq 0} P_{ir}^+(t)\, {}_0M_{rj}(+\infty)$ has a finite limit for all $j \in I^+$, that is, $M_{00}(t)\, {}_0M_{0j}(+\infty) - M_{0j}(t)$ has a finite limit for all $j \in I^+$. Conditions under which this limit exists will be investigated in Section 6.

4. Potentials for a Continuous Parameter Markov Chain

In this section, several results from the last section are applied to MRP's which are, in fact, c.p.M.C.'s. It is assumed that the Z-process is a c.p.M.C., that $p_{ii} = 0$, that $0 < q_i < +\infty$, and that $\sum_j q_{ij} = 0$ for all $i \in I^+$. In the notation of [12], the last two assumptions mean that each state is stable and that the process is conservative. Using the notation of Section 3, $\eta_i = -q_{ii}^{-1}$, $p_{ij} = \eta_i q_{ij}$ $(i \neq j)$, $0 < \eta_i < +\infty$, $\sum_j p_{ij} = 1$, $H_i(t) = 1 - e^{-q_i t}$, and $Q(t) = \mathscr{H}(t)\, P$. The following lemma gives several useful results which show how the notation simplifies if the MRP is a c.p.M.C.

Lemma 4.1. *For a recurrent, irreducible c.p.M.C. satisfying hypothesis A and for all $i, j \in I^+$ and $t \geqslant 0$,*

$$\eta_i H_i(t) = \bar{H}_i(t). \tag{4.1}$$

$$\mathscr{P}^+(t) = \mathscr{P}(t)\, P. \tag{4.2}$$

$$\sum_{r\neq 0} P_{ir}^+(t)\, {}_0\bar{P}_{rj}(+\infty) = \sum_r P_{ir}(t)\, {}_0\bar{P}_{rj}(+\infty) - \eta_j P_{ij}(t). \tag{4.3}$$

$$M_{ij} * D_j(t) = \eta_j P_{ij}(t). \tag{4.4}$$

$${}_0A_{ij}(t) = \sum_r P_{ir}(t)\, {}_0\bar{P}_{rj}(+\infty). \tag{4.5}$$

Proof. (4.1). This is clear. That (4.2) actually holds for any MRP for which $Q(t) = \mathscr{H}(t)\, P$ is shown by the following calculation.

$$\begin{aligned}\mathscr{P}^+(t) &= \mathscr{M}*(P - \mathscr{Q})\,(t)\\ &= \mathscr{M}*(I - \mathscr{H})\,(t)\, P\\ &= \mathscr{P}(t)\, P.\end{aligned}$$

By (4.2),

$$\sum_{r\neq 0} P^{+}_{ir}(t)\,{}_0\bar{P}_{rj}(+\infty) = \sum_{r\neq 0}\sum_{m} P_{im}(t)\,p_{mr}\,{}_0\bar{P}_{rj}(+\infty)$$

$$= \sum_{m} P_{im}(t) \sum_{r\neq 0} p_{mr}\,{}_0\bar{P}_{rj}(+\infty).$$

From 2 in the proof of Lemma 3.3,

$$\sum_{r\neq 0} p_{mr}\,{}_0\bar{P}_{rj}(+\infty) = {}_0\bar{P}_{mj}(+\infty) - \delta_{mj}\eta_m$$

so that

$$\sum_{r\neq 0} P^{+}_{ir}(t)\,{}_0\bar{P}_{rj}(+\infty) = \sum_{m} P_{im}(t)\,{}_0\bar{P}_{mj}(+\infty) - \eta_j P_{ij}(t).$$

$$D_j(t) = \int_t^{+\infty} [1 - H_j(u)]\,du = \eta_j\,e^{-q_j t} = \eta_j[1 - H_j(t)]$$

and

$$M_{ij} * D_j(t) = \eta_j P_{ij}(t).$$

Equation (4.5) follows from (4.3) and (4.4) and the definition of ${}_0A_{ij}(t)$.

The assumptions made in Section 3 that $\sigma_j^2 < +\infty$ and that $\lim_{t\to+\infty} M_{jj} * D_j(t)$ exist for all $j \in I^+$ are automatically satisfied by c.p.M.C.'s since $\sigma_j^2 = \eta_j^2$ and $\lim_{t\to+\infty} M_{jj} * D_j(t) = \eta_j \lim_{t\to+\infty} P_{ij}(t) = \eta_j^2\mu_{jj}^{-1}$ (where the last quantity is interpreted as zero if $\mu_{jj} = +\infty$). Hence, all of the results of Section 3 are true for c.p.M.C.'s. In particular, $|\,{}_0A_{ij}(t)\,| \leqslant {}_0\bar{P}_{jj}(+\infty)$ and a sufficient condition for the potential of f to exist is that $\sum_j {}_0\bar{P}_{jj}(+\infty)\,|\,f_j\,| < +\infty$. Further, all positive recurrent c.p.M.C.'s are left-normal and a null-recurrent c.p.M.C. is left-normal if and only if $\sum_r P_{ir}(t)\,{}_0M_{rj}(+\infty)$ has a finite limit for all $j \in I^+$.

The results of Section 3 can also be applied to c.p.M.C.'s to obtain results for a potential of measures, defined as follows.

DEFINITION 4.1. A σ-finite signed measure ν defined on I^+ is said to be a *potential* if there exists a signed measure μ defined on I^+ such that $|\,\mu\,|\,\mathbf{1} < +\infty$ and such that $\nu = \lim_{t\to+\infty} \mu\mathscr{P}(t)$.

The means by which Section 3 may be applied is the reversed process which was discussed in [10], and the principle of duality which was first used by Hunt in [13] and later by Kemeny and Snell in [3]. Set $N = (m_i\delta_{ij})$. It is clear that for any irreducible, recurrent, nonlattice c.p.M.C. satisfying hypothesis A, the reversed process is again an irreducible, recurrent, nonlattice, c.p.M.C. satisfying hypothesis A, for which

$$P^* = N^{-1}P'N, \qquad Q^*(t) = \mathscr{H}(t)\,P^*, \qquad \mathscr{M}^*(t) = N^{-1}\eta\mathscr{H}*\mathscr{M}(t)'P'N - I,$$

$$\bar{\mathscr{P}}^*(t) = N^{-1}\eta^{-1}\bar{\mathscr{P}}(t)'\,\eta N, \qquad \text{and} \qquad {}_0\bar{P}^*_{ij}(+\infty) = \pi_j\pi_i^{-1}\,{}_0\bar{P}_{ji}(+\infty)$$

if $i \neq 0$ while ${}_0\bar{P}^*_{0j}(+\infty) = {}_0\bar{P}_{0j}(+\infty)$. The mapping of functions given by $f^* = f'\eta N$ is 1–1 and onto, with the inverse mapping from measures to functions given by $\nu^* = N^{-1}\eta^{-1}\nu'$. It is clear that $\nu^* = \lim_{t\to+\infty}\bar{\mathscr{P}}^*(t)\,\mu^*$ exists if and only if $\nu = \lim_{t\to+\infty}\mu\bar{\mathscr{P}}(t)$ exists so that all results for potentials of functions may be used to obtain results for potentials of measures. These results are summarized below.

THEOREM 4.1. *If ν is a potential arising from μ, then $\mu 1 = 0$ and*

$$\nu = \mu_0\bar{\mathscr{P}}(+\infty) + \pi_0^{-1}\nu_0\pi. \tag{4.6}$$

Set

$$C_{ij} = \lim_{t\to+\infty}[\eta_j M_{jj}(t) - \bar{P}_{ij}(t)]$$

whenever it exists.

DEFINITION 4.2. Whenever it exists, $C = (C_{ij})$ will be called a *potential kernel* for the Z-process.

DEFINITION 4.3. A c.p.M.C. is said to be *right-normal* if

$${}_0\nu_j^* = \lim_{t\to+\infty} {}_0A^*_{ij}(t)$$

exists for all $j \in I^+$ and some state, 0. A c.p.M.C. is said to be *normal* if it is both left- and right-normal.

THEOREM 4.2. *${}_0\nu_j^*$ exists for all $j \in I^+$ if and only if C_{j0} exists for all $j \in I^+$. In this case ${}_i\nu_j^*$ exists for all $i, j \in I^+$ and ${}_i\nu_j^* = \pi_j\pi_i^{-1}C_{ji}$.*

PROOF.

$$\begin{aligned} {}_0A^*_{0j}(t) &= M^*_{00}(t)\,{}_0\bar{P}^*_{0j}(+\infty) - \bar{P}^*_{0j}(t) \\ &= M_{00}(t)\,{}_0\bar{P}_{0j}(+\infty) - \pi_j\pi_0^{-1}\bar{P}_{j0}(t) \\ &= \pi_j\pi_0^{-1}[\eta_0 M_{00}(t) - \bar{P}_{j0}(t)] \end{aligned}$$

which proves the first statement. The second follows as in Theorem 3.3.

THEOREM 4.3. *If the reversible c.p.M.C. is right-normal and μ is such that $\sum_{j} {}_jM_{00}(+\infty) \mid \mu_j \mid < +\infty$, then ν exists and $\nu = -\mu C$.*

The fact that $\bar{\mathscr{P}}^*(t) = N^{-1}\eta^{-1}\bar{\mathscr{P}}(t)'\,\eta N$, so that potential functions are mapped onto potential measures and vice-versa, is true for a wider class of MRP's than just c.p.M.C.'s as is shown in the following

LEMMA 4.2. *Let an MRP be such that $Q(t) = \mathscr{H}(t)\,P$. Then for all $t \geqslant 0$*

$$\mathscr{P}^*(t) = N^{-1}\eta^{-1}\mathscr{P}(t)'\,\eta N \tag{4.1}$$

if and only if $\eta_i H_i * \tilde{H}_j(t) = \eta_j H_j * \tilde{H}_i(t)$ for all $i, j, \in I^+$ and $t \geqslant 0$.

PROOF. Suppose $\mathscr{P}^*(t) = N^{-1}\eta^{-1}\mathscr{P}(t)'\,\eta N$. Then

$$(\mathscr{H} * \mathscr{M}'P' + I) * \tilde{\mathscr{H}}(t) = \eta^{-1}\bar{\mathscr{P}}(t)'\,\eta.$$

Convoluting by $\mathscr{H}(t)$ on the right of each side of the above equation yields

$$(\mathscr{H} * \mathscr{M}' * Q' + \mathscr{H}) * \tilde{\mathscr{H}}(t) = \eta^{-1}\tilde{\mathscr{H}} * \mathscr{M}' * \mathscr{H}(t)\,\eta$$

or

$$\eta\mathscr{H} * \mathscr{M}' * \tilde{\mathscr{H}}(t) = \tilde{\mathscr{H}} * \mathscr{M}' * \mathscr{H}(t)\,\eta$$

since $Q * \mathscr{M}(t) = \mathscr{M}(t) - I$. Taking Laplace-Stieltjes transforms, the above equation becomes

$$\eta_i h_i(s)\, m_{ji}(s)\, \tilde{h}_j(s) = \eta_j h_j(s)\, \tilde{h}_i(s)\, m_{ji}(s)$$

or

$$\eta_i h_i(s)\, \tilde{h}_j(s) = \eta_j h_j(s)\, \tilde{h}_i(s).$$

It follows that $\eta_i H_i * \tilde{H}_j(t) = \eta_j H_j * \tilde{H}_i(t)$ by the uniqueness of the Laplace-Stieltjes transform.

If the condition holds, then

$$\begin{aligned}
\bar{P}^*_{ij}(t) &= m_j m_i^{-1} H_i * \sum_k p_{jk} M_{ki} * \tilde{H}_j(t) + \delta_{ij}\tilde{H}_j(t) \\
&= m_j m_i^{-1}\eta_j\eta_i^{-1} H_j * \tilde{H}_i * \sum_k p_{jk} M_{ki}(t) + \delta_{ij}\tilde{H}_j(t) \\
&= \pi_j\pi_i^{-1}(M_{ji} - \delta_{ji}) * \tilde{H}_i(t) + \delta_{ij}\tilde{H}_j(t) \\
&= \pi_j\pi_i^{-1}\bar{P}_{ji}(t)
\end{aligned}$$

and

$$\mathscr{P}^*(t) = N^{-1}\eta^{-1}\bar{\mathscr{P}}(t)'\,\eta N.$$

This lemma shows that duality cannot be used for all MRP's to obtain results on the potentials of measures from the corresponding results for functions.

5. Potentials for the (Z, Z^+, V)-Process

Let $W_t = (Z_t, Z_t^+, V_t)$ for $t \geqslant 0$. In this section, a potential is defined, with respect to the W-process, for real-valued functions defined on $\mathfrak{X} = I^+ \times I^+ \times [0, +\infty)$. The W-process is a Markov process whose transition function

$$Q_t(i, j, x; k, m, y) = P[Z_t = k, Z_t^+ = m, V_t \leqslant y \mid Z_0 = i, Z_0^+ = j, V_0 = x]$$

can easily be calculated as

$$Q_t(i, j, x; k, m, y) = \begin{cases} R_j(k, m, y; t - x) & x \leqslant t \\ \delta_{ik}\delta_{jm} & t < x \leqslant t + y \\ 0 & t + y \leqslant x. \end{cases} \tag{5.1}$$

It is assumed that $\lim_{t\to+\infty} M_{jj} * D_{jk}(y, \cdot)\,(t)$ exists finitely for all $j, k \in I^+$ and $y \geqslant 0$ for the rest of this section. Many of the results obtained are similar to those in Section 3 and where the proofs are similar, they are omitted. If ν is a measure on $\mathfrak{X}$, set

$$\nu \circ f = \sum_{k,m} \int_0^\infty f(k, m, y)\, \nu(k, m, dy).$$

Let

$$\pi(i, j, x) = m_i p_{ij} \int_0^x [1 - F_{ij}(u)]\, du \qquad \text{and} \qquad \pi_1(i, j, x) = m_i^{-1} \pi(i, j, x).$$

Again, fix 0 as a reference state.

Definition 5.1. A real-valued function, g, defined on $\mathfrak{X}$ is said to be a *potential* if there exists a real-valued function, f, defined on $\mathfrak{X}$, such that $\pi \circ |f| < +\infty$, $\pi \circ f = 0$ and such that

$$g(i, j, x) = \lim_{t\to+\infty} \bar{Q}_t(i, j, x; \cdot) \circ f.$$

Using (5.1), it is seen that for $t > x$,

$$\bar{Q}_t(i, j, x; k, m, y) = \delta_{ik}\delta_{jm} \min(x, y) + \bar{R}_j(k, m, y; t - x)$$

and

$$\bar{Q}_t(i, j, x; \cdot) \circ f = \int_0^x f(i, j, y)\, dy + \bar{R}_j(\cdot\,; t - x) \circ f.$$

From the above equation, it is seen that a potential g, if it exists, consists of two terms. One of these is the Lebesgue integral of f with respect to the third variable and this term is independent of t while the other term is independent of i and x. In order that g exist, it will be necessary that $\int_0^x f(i, j, y)\, dy < +\infty$ for all $i, j \in I^+$ and $x \geqslant 0$. The major concern, then, is with the second term which depends on t. The condition that $\pi \circ |f| < +\infty$ is equivalent to $\sum_k m_k |f_k| < +\infty$ and the condition that $\pi \circ f = 0$ is equivalent to $\sum_k m_k f_k = 0$ where

$$|f_k| = \sum_m \int_{0-}^{+\infty} |f(k, m, y)|\, \pi_1(k, m, dy)$$

and

$$f_k = \sum_m \int_{0-}^{+\infty} f(k, m, y)\, \pi_1(k, m, dy).$$

Whenever g exists, set

$$g^*(i, j, x) = g(i, j, x) - \int_0^x f(i, j, y)\, dy.$$

LEMMA 5.1. *If* $f = (f_0, f_1, \cdots)'$ *and* $\sum_k m_k f_k = 0$, *then*

$$f = (I - P)_0 \mathscr{M}(+\infty) f.$$

PROOF. See Lemma 3.3.

LEMMA 5.2. *If* $\pi \circ |f| < +\infty$, *then* ${}_0\bar{R}_i(\cdot; +) \circ |f| < +\infty$.

PROOF. ${}_0\bar{R}_i(j, m, y; +\infty) = {}_0M_{ij}(+\infty)\, \pi_1(j, m, y)$. Hence,

$$\begin{aligned} {}_0\bar{R}_i(\cdot; +\infty) \circ |f| &= \sum_{j,m} {}_0M_{ij}(+\infty) \int_{0-}^{+\infty} |f(j, m, y)|\, \pi_1(j, m, dy) \\ &= \sum_j {}_0M_{ij}(+\infty)\, |f_j| \\ &\leqslant \sum_j [{}_0M_{ij}(+\infty) + {}_iM_{0j}(+\infty)]\, |f_j| \\ &= {}_0M_{ii}(+\infty) \sum_j {}_iM_{ij}(+\infty)\, |f_j| < +\infty. \end{aligned}$$

LEMMA 5.3. *If* g *is a potential arising from* f, *then*

$$g^*(i, j, x) = {}_0\bar{R}_j(\cdot; +\infty) \circ f - \lim_{t \to +\infty} {}_0B_j(\cdot; t - x) \circ f.$$

PROOF.

$$M_{jm}(t)\,\pi_1(m,r,y) - \bar{R}_j(m,r,y;t) = M_{jm} * D_{mr}(y,\cdot)\,(t).$$

Hence,

$$\begin{aligned} R_j(\cdot;t)\circ f &= \sum_{m,r} M_{jm}(t)\int_{0-}^{+\infty} f(m,r,y)\,\pi_1(m,r,dy) \\ &\qquad - \sum_{m,r}\int_{0-}^{+\infty} f(m,r,y)\,M_{jm} * D_{mr}(dy,\cdot)\,(t) \\ &= \sum_m M_{jm}(t) f_m - W_j(t) \end{aligned}$$

where

$$W_j(t) = \sum_{m,r}\int_{0-}^{+\infty} f(m,r,y)\,M_{jm} * D_{mr}(dy,\cdot)\,(t).$$

By Lemma 5.1, (1.12.8), and (2.16), this becomes

$$\begin{aligned} [\mathcal{M}(t)\,(I-P)_0\,\mathcal{M}(+\infty)f]_j - W_j(t) &= [(I-\mathscr{P}^+(t))_0\,\mathcal{M}(+\infty)f]_j - W_j(t) \\ &= [{}_0\mathcal{M}(+\infty)f]_j - [\mathscr{P}^+(t)_0\mathcal{M}(+\infty)f]_j \\ &\qquad - W_j(t) \\ &= {}_0\bar{R}_j(\cdot;+\infty)\circ f - {}_0B_j(\cdot;t)\circ f. \end{aligned}$$

Set

$$B_i(j,m,y) = \lim_{t\to+\infty}\,[{}_i\bar{R}_i(j,m,y;+\infty)\,M_{ii}(t) - \bar{R}_i(j,m,y;t)]$$

whenever this limit exists.

DEFINITION 5.2. Whenever this limit exists for all j, $m \in I^+$ and $y \geqslant 0$ and all states i, $B_i(j,m,y)$ will be called a *potential kernel* with respect to the W-process.

DEFINITION 5.3. The three-dimensional W-process is said to be *left-normal* if

$${}_0\nu(j,m,y) = \lim_{t\to+\infty}\,{}_0B_i(j,m,y;t)$$

exists for all j, $m \in I^+$, $y \geqslant 0$, and some state 0.

Lemma 2.7 shows that under the assumption made at the beginning of this section, the limit is independent of the initial state, i. The following theorem shows that left-normality is equivalent to the existence of the potential kernel.

THEOREM 5.1. *${}_0\nu(j,m,y)$ exists for all j, $m \in I^+$ and $y \geqslant 0$ if and only if $B_0(j,m,y)$ exists for all j, $m \in I^+$ and $y \geqslant 0$. In this case, ${}_i\nu(\alpha,\beta,x)$ exists for all i, $\alpha,\beta \in I^+$ and $x \geqslant 0$ and ${}_i\nu(\alpha,\beta,x) = B_i(\alpha,\beta,x)$.*

PROOF. See Theorem 3.3.

COROLLARY 5.1. *Under the hypotheses,*

$$\begin{aligned} &B_i(\alpha, \beta, x) - \pi(\alpha, \beta, x)\, \pi(i, j, y)^{-1}\, B_i(i, j, y) \\ &= {}_0\nu(\alpha, \beta, x)\, {}_0R_i(i, j, y)\, \pi(\alpha, \beta, x)\, \pi(i, j, y)^{-1} - {}_0R_i(\alpha, \beta, x) \\ &\qquad - {}_0\nu(i, j, y)\, \pi(\alpha, \beta, x)\, \pi(i, j, y)^{-1}. \end{aligned}$$

The following theorem gives a sufficient condition for a potential to exist and be expressible in terms of the potential kernel defined in Definition 5.2.

THEOREM 5.2. *If the W-process associated with an MRP is left-normal and f is a real-valued function defined on $\mathfrak{X}$ such that $\pi \circ |f| < +\infty$, $\pi \circ f = 0$, $\int_0^\infty f(i, j, y)\, dy < +\infty$ for all $i, j \in I^+$ and $x \geqslant 0$, and*

$$\lim_{t \to +\infty} {}_0B(\cdot\,; t) \circ f = {}_0\nu \circ f < +\infty,$$

then g^* exists and $g^*(i, j, y) = -B_j \circ f$.

PROOF. See Theorem 3.4.

COROLLARY 5.2. *Under the above hypotheses,*

$$g(i, j, y) = \int_0^y f(i, j, u)\, du - B_j \circ f.$$

COROLLARY 5.3. *Set*

$$A(i, j, y; k, m, x) = \delta_{ik}\delta_{jm} \min(x, y) - B_j(k, m, x).$$

Under the above hypotheses, g exists, and $g(i, j, y) = A(i, j, y; \cdot) \circ f$.

Turning to the positive recurrent case, the following theorem gives the analogous result to Theorem 3.5.

THEOREM 5.3. *The W-process associated with a positive recurrent MRP is left-normal and $b_{ij} < +\infty$ for all $i, j \in I^+$.*

In the null-recurrent case, the W-process is left-normal if and only if $\sum_{r \neq 0} P_{ir}^+(t)\, {}_0M_{rj}(+\infty)$ has a finite limit for all $j \in I^+$. This seems to be the same condition as in Section 3. The difference is that in Section 3, it was assumed that $\lim_{t \to +\infty} M_{jj} * D_j(t) < +\infty$ whereas, in this section, it is assumed only that $\lim_{t \to +\infty} M_{jj} * D_{jm}(y, \cdot)(t) < +\infty$. This is a weaker condition so that the W-process may be left-normal when the Z-process is not. Conditions under which the assumption made at the beginning of this section is true are studied in the next section.

6. Miscellaneous Results

In Section 3, it was assumed that $\lim_{t\to+\infty} M_{jj} * D_j(t) < +\infty$ for all $j \in I^+$ and in Section 5, it was assumed that $\lim_{t\to+\infty} M_{jj} * D_{jm}(y, \cdot)(t) < +\infty$ for all $j \in I^+$ and $y \geqslant 0$. If $\int_0^{+\infty} D_j(u)\, du < +\infty$ or $\int_0^{+\infty} D_{jm}(y, u)\, du < +\infty$, the Key Renewal theorem shows that these limits exist. The question remains as to when these limits exist if $\mu_{jj} = +\infty$ and $\int_0^{+\infty} D_j(u)\, du = +\infty$ or $\int_0^{+\infty} D_{jm}(y, u)\, du = +\infty$. The following theorems give a partial answer to this question.

Theorem 6.1. *If*

$$\lim_{t\to+\infty} D_j(t)\,[1 - G_{jj}(t)]^{-1} = A < +\infty, \qquad \textit{then} \qquad \lim_{t\to+\infty} M_{jj} * D_j(t) = A$$

Similarly, if

$$\lim_{t\to+\infty} D_{jm}(y, t)\,[1 - G_{jj}(t)]^{-1} = A$$

for any fixed $y \geqslant 0$, *then*

$$\lim_{t\to+\infty} M_{jj} * D_{jm}(y, \cdot)\,(t) = A.$$

Proof. Choose $\epsilon > 0$. By hypothesis, there is a t_0 such that $t \geqslant t_0$ implies that $|\, D_j(t)\,[1 - G_{jj}(t)]^{-1} - A \,| < \epsilon$. Then for $t \geqslant t_0$,

$$\begin{aligned} D_j * M_{jj}(t) &= \int_{0-}^{t-t_0} D_j(t-u)\, dM_{jj}(u) + \int_{t-t_0}^{t} D_j(t-u)\, dM_{jj}(u) \\ &\leqslant \int_{0-}^{t-t_0} (A+\epsilon)\,[1 - G_{jj}(t-u)]\, dM_{jj}(u) \\ &\qquad\qquad + \eta_j[M_{jj}(t) - M_{jj}(t-t_0)] \\ &\leqslant (A+\epsilon) \int_{0-}^{t} [1 - G_{jj}(t-u)]\, dM_{jj}(u) \\ &\qquad\qquad + \eta_j[M_{jj}(t) - M_{jj}(t-t_0)] \end{aligned}$$

Hence, $\overline{\lim}_{t\to+\infty} D_j * M_{jj}(t) \leqslant A + \epsilon$ since $[M_{jj}(t) - M_{jj}(t - t_0)] \to 0$ as $t \to +\infty$ by the Key Renewal theorem and $(1 - G_{jj}) * M_{jj}(t) \equiv 1$. If $A = 0$, the proof is complete since $D_j * M_{jj}(t) \geqslant 0$ and ϵ is arbitrary. Otherwise, it may similarly be shown that $\underline{\lim}_{t\to+\infty} D_j * M_{jj}(t) \geqslant A - \epsilon$ and since ϵ is arbitrary, the proof is complete. The second statement is proved in exactly the same way.

Theorem 6.2. *If*

$$\lim_{t\to+\infty} D_j * M_{jj}(t) = A < +\infty, \qquad \textit{and} \qquad |\, D_j * M_{jj}(t) \,| \leqslant B$$

for some fixed constant B, then $\lim_{t\to+\infty} \bar{D}_j(t)\, \tilde{G}_{jj}(t)^{-1} = A$. *Similarly, if*

$$\lim_{t\to+\infty} D_{jm}(y, \cdot) * M_{jj}(t) = A < +\infty, \qquad \text{and} \qquad | D_{jm}(y, \cdot) * M_{jj}(t) | \leqslant B$$

for some fixed constant B, then

$$\lim_{t\to+\infty} \bar{D}_{jm}(y, t)\, \tilde{G}_{jj}(t)^{-1} = A.$$

PROOF. Let $K_j(t) = D_j * M_{jj}(t)$. Then $K_j * (1 - G_{jj})\,(t) = D_j(t)$ so that

$$K_j * \tilde{G}_{jj}(t)\, \tilde{G}_{jj}(t)^{-1} = \bar{D}_j(t)\, \tilde{G}_{jj}(t)^{-1}.$$

It is a standard Abelian argument to show that $K_j(+\infty) = A$ and $| K_j(t) | \leqslant B$ imply that $\lim_{t\to+\infty} \bar{D}_j(t)\, \tilde{G}_{jj}(t)^{-1} = A$. The second statement is proved similarly.

In general, $\lim_{t\to+\infty} \bar{D}_j(t)\, \tilde{G}_{jj}(t)^{-1} = A$ does not imply that

$$\lim_{t\to+\infty} D_j(t)\, [(1 - G_{jj})\,(t)]^{-1} = A.$$

But this implication is true if $1 - G_{jj}(x) \sim x^{-\alpha} G_j(x)$ as $x \to +\infty$ where $0 < \alpha < 1$ and where G_j is a slowly varying function.

THEOREM 6.3. *If* $1 - G_{jj}(x) \sim x^{-\alpha} G_j(x)$ *as* $x \to +\infty$ *where* $0 < \alpha < 1$ *and where* G_j *is a slowly varying function, if* $| D_j * M_{jj}(t) | \leqslant B$, *and if*

$| D_{jm}(y, \cdot) * M_{jj}(t) | \leqslant C$, *then* $\lim_{t\to+\infty} D_j * M_{jj}(t) = A > 0$ *if and only if*

$$\lim_{t\to+\infty} D_j(t)\, [(1 - G_{jj})\,(t)]^{-1} = A, \qquad \textit{and} \qquad \lim_{t\to+\infty} D_{jm}(y, \cdot) * M_{jj}(t) = A > 0$$

if and only if

$$\lim_{t\to+\infty} D_{jm}(y, t)\, [(1 - G_{jj})\,(t)]^{-1} = A.$$

PROOF. After applying Theorems 6.1 and 6.2, all that remains is to show that

$$\lim_{t\to+\infty} \bar{D}_j(t)\, \tilde{G}_{jj}(t)^{-1} = A$$

implies that

$$\lim_{t\to+\infty} D_j(t)\, [(1 - G_{jj})\,(t)]^{-1} = A.$$

In [14], Lamperti shows that $1 - G_{jj}(t) \sim t^{-\alpha} G_j(t)$ implies that

$$\lim_{t\to+\infty} [1 - G_{jj}(t)]\, \tilde{G}_{jj}(t)^{-1} = (1 - \alpha)$$

or that $\tilde{G}_{jj}(t) \sim (1 - \alpha)\, t^{1-\alpha} G_j(t)$. Hence, $\bar{D}_j(t) \sim A(1 - \alpha)\, t^{1-\alpha} G_j(t)$. By applying Theorem 2 of [15], it is seen that $D_j(t) \sim A t^{-\alpha} G_j(t)$ and hence that

$$\lim_{t \to +\infty} D_j(t)\, [(1 - G_{jj})\,(t)]^{-1} = A.$$

The second statement is proved in exactly the same way.

The above theorems show that

$$\lim_{t \to +\infty} D_j * M_{jj}(t) < +\infty \qquad \text{if} \qquad \lim_{t \to +\infty} D_j(t)\,(1 - G_{jj})\,(t)^{-1} < +\infty$$

and that

$$\lim_{t \to +\infty} D_{jm}(y, \cdot) * M_{jj}(t) < +\infty \quad \text{if} \quad \lim_{t \to +\infty} D_{jm}(y, t)\,(1 - G_{jj})\,(t)^{-1} < +\infty.$$

If the random variable which records the length of time between visits to j is in the domain of attraction of an α-stable law ($0 < \alpha < 1$), and if each quantity is bounded, then each limit is finite if and only if the corresponding ratio has a finite limit.

In the case of a null-recurrent MRP, it was seen that the MRP was left-normal if and only if

$$\lim_{t \to +\infty} [M_{00}(t)\, {}_0M_{0j}(+\infty) - M_{0j}(t)] < +\infty$$

for all $j \in I^+$. In [9] it is shown that

$$\begin{aligned} &M_{00}(t)\, {}_0M_{0j}(+\infty) - M_{0j}(t) \\ &= {}_0M_{jj}(+\infty)\,\{[{}_0G_{jj}(+\infty) - {}_0G_{jj}] * {}_0M_{0j} + [{}_0G_{0j}(+\infty) - {}_0G_{0j}]\} * M_{00}(t). \end{aligned}$$

Let

$$H_{0j}(t) = [{}_0G_{jj}(+\infty) - {}_0G_{jj}] * {}_0M_{0j}(t) + {}_0G_{0j}(+\infty) - {}_0G_{0j}(t).$$

Then an argument similar to that used in Theorem 6.1 may be applied with the result that ${}_0\nu_j$ exists if

$$\lim_{t \to +\infty} H_{0j}(t)\,[1 - G_{00}(t)]^{-1} < +\infty.$$

By an argument similar to that used in Theorem 6.3, ${}_0\nu_j$ exists if and only if the above limit exists, whenever the first passage time random variable for 0 is in the domain of attraction of an α-stable law since

$$|\, M_{00}(t)\, {}_0M_{0j}(+\infty) - M_{0j}(t)\,| \leqslant {}_0M_{jj}(+\infty) < +\infty.$$

In [4] and [5], Kemeny and Snell give an example showing that ${}_i\nu_j$ need not exist. In constructing this example, they make use of a result by Orey [16] for M.C.'s. This result generalizes directly to MRP's, thereby showing that ${}_i\nu_j$ need not exist for a null-recurrent MRP.

As before, a small letter stands for the Laplace-Stieltjes transform of the corresponding capital letter. The following relationships are easily verified.

$$ {}_kM_{jk}(t) = \delta_{jk}\,. \tag{6.1}$$

$$ {}_kM_{jj}(t) = {}_kM_{jj} * {}_kG_{jj}(t) + 1. \tag{6.2}$$

$$ {}_jM_{jk}(t) = {}_jM_{kk} * {}_jG_{jk}(t). \tag{6.3}$$

$$ G_{jj}(t) = {}_kG_{jj}(t) + G_{kj} * {}_jG_{jk}(t). \tag{6.4}$$

$$ G_{jk}(t) = {}_jG_{jk} * {}_kM_{jj}(t) = G_{jk} * {}_kG_{jj}(t) + {}_jG_{jk}(t). \tag{6.5}$$

$$ {}_jG_{jk}(+\infty) + {}_kG_{jj}(+\infty) = 1, \qquad {}_jG_{kk}(+\infty) + {}_kG_{kj}(+\infty) = 1. \tag{6.6}$$

From these, one obtains

$$ {}_km_{jj}(s) = [1 - {}_kg_{jj}(s)]^{-1}. \tag{6.7}$$

$$ {}_jm_{jk}(s) = {}_jm_{kk}(s)\,{}_jg_{jk}(s) = {}_jg_{jk}(s)\,[1 - {}_jg_{kk}(s)]^{-1}. \tag{6.8}$$

$$ g_{jj}(s) = {}_kg_{jj}(s) + g_{kj}(s)\,{}_jg_{jk}(s). \tag{6.9}$$

$$ g_{jk}(s) = {}_jg_{jk}(s)\,{}_km_{jj}(s). \tag{6.10}$$

$$ g_{jj}(s) = {}_kg_{jj}(s) + {}_jg_{jk}(s)\,{}_kg_{kj}(s)\,[1 - {}_jg_{kk}(s)]^{-1}. \tag{6.11}$$

$$ g_{jk}(s) = {}_jg_{jk}(s)\,[1 - {}_kg_{jj}(s)]^{-1}. \tag{6.12}$$

From the above relationships, it follows that

$$ \begin{aligned} {}_jG_{jk}(+\infty)\,{}_kG_{kj}(+\infty)^{-1} &= {}_jG_{jk}(+\infty)\,[1 - {}_jG_{kk}(+\infty)]^{-1} \\ &= {}_jm_{kk}(0)\,{}_jg_{jk}(0) = {}_jm_{jk}(0) = {}_jM_{jk}(+\infty). \end{aligned} $$

Set

$$ R_{jk}(t) = {}_kG_{kj}(+\infty)\,[\eta_k M_{kk}(t) - \bar{P}_{jk}(t)] $$

and

$$ T_{jk}(t) = {}_jG_{jk}(+\infty)\,\eta_k M_{jj}(t) - {}_kG_{kj}(+\infty)\,\bar{P}_{jk}(t). $$

Then

$$ T_{jk}(t) = {}_kG_{kj}(+\infty)\,[{}_j\bar{P}_{jk}(+\infty)\,M_{jj}(t) - \bar{P}_{jk}(t)] $$

and (ignoring constants), R_{jk} and T_{jk} are the functions from which the potential kernels discussed in Sections 3 and 4 arise.

Fix j and k. Let $*$ used as a superscript denote that the roles of the subscripts j and k have been reversed. Following Orey, set

$$ \begin{aligned} a(s) &= 1 - {}_kg_{jj}(s) - {}_jg_{jk}(s). \\ b(s) &= 1 - {}_jg_{kk}(s) - {}_kg_{kj}(s). \\ c(s) &= {}_kg_{kj}(s)\,[1 - {}_kg_{jj}(s)] - {}_jg_{jk}(s)\,[1 - {}_jg_{kk}(s)]. \\ d(s) &= [1 - {}_kg_{jj}(s)]\,[1 - {}_jg_{kk}(s)] - {}_jg_{jk}(s)\,{}_kg_{kj}(s). \end{aligned} $$

The relationships given in the following lemma are easily derived using (6.1) through (6.12).

LEMMA 6.1. *For all* $s \geqslant 0$,

$$r_{jk}(s) = {}_kG_{kj}(+\infty)\,\eta_k a(s)\,d(s)^{-1} + \eta_k d(s)^{-1}\,{}_kG_{kj}(+\infty)\,{}_jg_{jk}(s)\,[1 - \tilde{h}_k(s)\,\eta_k^{-1}]. \tag{6.13}$$

$$t_{jk}(s) = \eta_k d(s)^{-1}\{{}_jG_{jk}(+\infty)\,[1 - {}_jg_{kk}(s)] - {}_kG_{kj}(+\infty)\,{}_jg_{jk}(s)\} + \eta_k d(s)^{-1}\,{}_kG_{kj}(+\infty)\,{}_jg_{jk}(s)\,[1 - \tilde{h}_k(s)\,\eta_k^{-1}]. \tag{6.14}$$

$${}_kG_{kj}(+\infty)\,m_{jk}(s)\,d_k(s) = \eta_k d(s)^{-1}\,{}_kG_{kj}(+\infty)\,{}_jg_{jk}(s)\,[1 - \tilde{h}_k(s)\,\eta_k^{-1}] \tag{6.15}$$

$$r_{jk}(s) + r_{kj}(s) = t_{jk}(s) + t_{kj}(s). \tag{6.16}$$

For any $\alpha, \beta, \Upsilon \in \{j, k\}$, set ${}_\alpha\epsilon_{\beta\Upsilon}(s) = {}_\alpha G_{\beta\Upsilon}(+\infty) - {}_\alpha g_{\beta\Upsilon}(s)$. Any ${}_\alpha\epsilon_{\beta\Upsilon}$ will be called an ϵ-term and any product of such terms will be called a higher order ϵ-term.

THEOREM 6.4. *As* $s \to 0$,

(1) $r_{jk}(s)$ *approaches a limit if and only if*

$$[{}_k\epsilon_{jj}(s) + {}_j\epsilon_{jk}(s)]\,[{}_j\epsilon_{kk}(s) + {}_k\epsilon_{kj}(s)]^{-1}$$

approaches a limit.

(2) $t_{jk}(s)$ *approaches a limit if and only if*

$$[{}_jG_{jk}(+\infty)\,{}_j\epsilon_{kk}(s) + {}_kG_{kj}(+\infty)\,{}_j\epsilon_{jk}(s)] \cdot [{}_jG_{jk}(+\infty)\,{}_k\epsilon_{kj}(s) + {}_kG_{kj}(+\infty)\,{}_k\epsilon_{jj}(s)]^{-1}$$

approaches a limit.

PROOF. Since, by assumption $M_{jj} * D_j(t)$ has a finite limit as $t \to +\infty$ for all $j \in I^+$, it follows that the left side of (6.15) has the same limit as $s \to 0$. Hence, r_{jk} and t_{jk} will have limits as $s \to 0$ if and only if the first terms on the right of (6.13) and (6.14) have limits.

Since

$$a(s) = {}_k\epsilon_{jj}(s) + {}_j\epsilon_{jk}(s)$$

and

$$d(s) = {}_jG_{jk}(+\infty)\,[{}_j\epsilon_{kk}(s) + {}_k\epsilon_{kj}(s)] + {}_kG_{kj}(+\infty)\,[{}_k\epsilon_{jj}(s) + {}_j\epsilon_{jk}(s)] + (\text{higher-order } \epsilon\text{-terms}),$$

the first result follows. Since

$$
\begin{aligned}
&{}_jG_{jk}(+\infty)\,[1 - {}_jg_{kk}(s)] - {}_kG_{kj}(+\infty)\,{}_jg_{jk}(s) \\
&\quad = {}_jG_{jk}(+\infty)\,{}_j\epsilon_{kk}(s) + {}_kG_{kj}(+\infty)\,{}_j\epsilon_{jk}(s),
\end{aligned}
$$

the second result follows. The proof is then complete.

This theorem shows that R_{jk} and T_{jk} need not always have a limit as $t \to +\infty$ since the ${}_\alpha\epsilon_{\beta\gamma}$ may be chosen so that r_{jk} and t_{jk} do not have a limit. Hence, the potential kernels defined in Definitions 3.3 and 4.2 do not always exist.

Although r_{jk} and t_{jk} do not always have a limit, the following theorem shows that $r_{jk} + r_{kj} = t_{jk} + t_{kj}$ does.

THEOREM 6.5. *As* $s \to 0$, $r_{jk}(s) + r_{kj}(s) \to 1 + K$ *where*

$$K = \lim_{t\to+\infty} [{}_kG_{kj}(+\infty)\, M_{jk} * D_k(t) + {}_jG_{jk}(+\infty)\, M_{kj} * D_j(t)].$$

PROOF. It is easily verified that

$$
\begin{aligned}
r_{jk}(s) + r_{kj}(s) = {}&\{{}_kG_{kj}(+\infty)\,\eta_k a(s)\, d(s)^{-1} + {}_jG_{jk}(+\infty)\,\eta_j a^*(s)\, d^*(s)^{-1}\} \\
&+ \{{}_kG_{kj}(+\infty)\, m_{jk}(s)\, d_k(s) + {}_jG_{jk}(+\infty)\, m_{kj}(s)\, d_j(s)\}.
\end{aligned}
$$

The second bracketed expression converges to K by the assumption made in Section 3. That the first term converges to 1, may be proved exactly as in Orey's proof in [16].

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